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The Behavior of Polyharmonic Cardinal Splines as Their Degree Tends to Infinity

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The behavior of cardinal splines interpolating elements of l_2^k as their degree tends to infinity in R^1 has been determined by I. J. Schoenberg in the 1970s. In this paper we consider the generalized problem in R^n by replacing cardinal splines by polyharmonic cardinal splines.

1. INTRODUCTION

The univariate cardinal spines were generalized into higher dimensional spaces by de Boor and DeVore in 1983. The box splines introduced in [1, 2] are believed to be a truly multivariate analogue of the cardinal splines. A few years later, W. Madych and S. Nelson generalized the unvariate cardinal splines in a different way. The polyharmonic cardinal splines introduced in [4] are believed to be another multivariate analogue of the cardinal splines. We use the generalization by W. Madych and S. Nelson here and consider the convergence of the interpolant with respect to certain data.

Let $f(x)$ be a function from R^n to C. If $f(x)$ has polynomial growth, then we can construct a function

$$S_k(f, x) = \sum_{\mathbf{j} \in Z^n} f(\mathbf{j})\, L_k(x - \mathbf{j})$$

which interpolates f at all the integer lattice points, where k is an integer such that $2k > n$ and $L_k(x)$ is the fundamental spline defined in [4]. Here we investigate the conditions under which $S_k(f, x)$ converges to f as k goes to infinity. We devote ourselves to this problem in this paper. In Section 2, we give definitions and some basic results about polyharmonic cardinal splines which are mostly based on [4, 5]. In Section 3, we will study

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the interpolatory properties of certain analytic functions. In Section 4, where the main results are, we examine the convergence properties of polyharmonic cardinal splines.

2. Notation, Definition, and Basic Results

We use the standard mathematical notations for the calculus in n variables, see [3]. By ν and μ, we denote the multi-indices, that is, n-tuples $(\nu_1, ..., \nu_n)$ and $(\mu_1, ..., \mu_n)$ of non-negative integers. The sum $\sum_{j=1}^{n} \nu_j$ will be denoted by $|\nu|$. Throughout the paper, k is a given integer, so that $2k \geqslant n+1$ and $x=(x_1, ..., x_n)$. For the sake of convenience, we set

$$D^\nu = \frac{\partial^{|\nu|}}{\partial^{\nu_1} x_1 \cdots \partial^{\nu_n} x_n}$$

and

$$x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}.$$

We denote the set of integers by Z; Z^n denotes the integer lattice in R^n and $Q^n = (-\pi, \pi)^n$. Elements of Z^n are denoted by boldface symbols such as $\mathbf{j}$. π_k represents the set of polynomials of degree less than or equal to k.

Definition 1. Given a sequence $v = \{v_{\mathbf{j}}\}$, $\mathbf{j} \in Z^n$, we say that $v \in l_2^k(Z^n)$ if

$$\|v\|_{l_2^k}^2 = \sum_{\mathbf{j} \in Z^n} \sum_{|\nu| = k} |T^\nu v_{\mathbf{j}}|^2 < \infty,$$

where $T^\nu = T_1^{\nu_1} T_2^{\nu_2} \cdots T_n^{\nu_n}$, $(T_i v)_{\mathbf{j}} = v_{\mathbf{j}+e_i} - v_{\mathbf{j}}$, and

$$e_i = (\overbrace{0, ..., 0}^{i-1}, 1, 0, ..., 0),$$

i.e., T_i is a difference operator in the direction e_i, $i = 1, ..., n$.

It is easy to see from the definition $\|v\|_{l_2^{k+1}} \leqslant 2\,\|v\|_{l_2^k}$. This includes $l_2^k \subset l_2^{k+1}$. We call this the monotone property of l_2^k. Recall that every element of l_2^k in R^1 is of polynomial growth of order k. This leads to the following proposition.

Proposition 1. *Every element of $l_2^k(Z^n)$ is of polynomial growth of order k.*

Proof. The proofs in different dimensions are closely related. We proof the result in R^1 first and use the same idea to prove the general result by induction. Suppose $v=(v_j)\in l_2^k(R^1)$, i.e.,

$$\sum_j |T^k v_j|^2 < \infty.$$

There exists a constant C_0 independent of j such that

$$|T^k v_j| \leqslant C_0, \qquad \text{for all} \quad j\in Z. \tag{1}$$

We claim the following is true:

$$|T^{k-i} v_j| \leqslant C_{k,1}(1+|j|)^i, \qquad j\in Z,\ i=0, ..., k. \tag{2}$$

If $i=0$, let $C_{k,1}=C_0$, then (2) reduces to (1). Suppose that (2) holds for $i-1$, that is,

$$|T^{k-(i-1)} v_j| \leqslant C_{k,1}(1+|j|)^{i-1}, \qquad j\in Z. \tag{3}$$

Observe

$$T^{k-i}v_j = T^{k-i}v_0 + \sum_{l=0}^{j-1} T^{k-i+1}v_l. \tag{4}$$

This implies

$$|T^{k-i}v_j| \leqslant |T^{k-i}v_0| + \sum_{l=0}^{j-1} |T^{k-i+1}v_l| \leqslant |T^{k-i}v_0| + C_{k,1}(1+|j|)^i$$
$$\leqslant C'_{k,1}(1+|j|)^i,$$

where $C'_{k,1}=\max_{i=0,...,k}\{C_{k,1}, |T^{k-i}v_0|\}$.

This concludes the proof of the claim. Let $i=k$, then we have

$$|v_j| \leqslant C_{k,1}(1+|j|)^k.$$

Thus the proposition holds in R^1.

Suppose the proposition holds in R^{n-1}, that is, if $\mathbf{j}_1=(j_2, ..., j_n)$ and $v=(v_{\mathbf{j}_1})\in l_2^i(Z^{n-1})$, $i=0, 1, ..., k$, then

$$|v_{\mathbf{j}_1}| \leqslant C_{k,n-1}(1+|\mathbf{j}_1|)^i. \tag{5}$$

Similarly, we claim that the relation

$$|T_1^{k-i}v_{\mathbf{j}}| \leqslant C_{k,n}(1+|\mathbf{j}|)^i \tag{6}$$

holds for $i=0, 1, ..., k$.

If $i=0$, inequality (6) holds by the definition of $l_2^k(Z^n)$. Let us suppose inequality (6) holds for $i-1$, i.e.,

$$|T_1^{k-i+1}v_{\mathbf{j}}| \leqslant C_{k,n}(1+|\mathbf{j}|)^{i-1}. \tag{7}$$

Recall from (4)

$$T_1^{k-i}v_{\mathbf{j}} = T_1^{k-i}v_{0,\mathbf{j}_1} + T_1^{k-i+1}v_{1,\mathbf{j}_1} + \cdots + T_1^{k-i+1}v_{j_1-1,\mathbf{j}_1},$$

and

$$T_1^{k-i}v_{0,\mathbf{j}_1} \in l_2^i(Z^{n-1}), \qquad i=0, ..., k.$$

We have

$$\begin{aligned} |T_1^{k-i}v_{\mathbf{j}}| &\leqslant |T_1^{k-i}v_{0,\mathbf{j}_1}| + |T_1^{k-i+1}v_{1,\mathbf{j}_1}| + \cdots + |T_1^{k-i+1}v_{j_1-1,\mathbf{j}_1}| \\ &\leqslant |T_1^{k-i}v_{0,\mathbf{j}_1}| + C_{k,n-1}\, |j_1|\, (1+|\mathbf{j}_1|)^{i-1}| \\ &\leqslant C_{i,n-1}(1+|\mathbf{j}_1|)^i + C_{k,n-1}(1+|\mathbf{j}|)^i \\ &\leqslant C_{i,n}(1+|\mathbf{j}|)^i, \end{aligned}$$

where $C_{k,n} = 2\max_{i=0,...,k}\{C_{i,n-1}\}$. So the claim holds.
Let $i=k$, then inequality (6) becomes

$$|v_{\mathbf{j}}| \leqslant C_{k,n}(1+|\mathbf{j}|)^k. \quad \blacksquare$$

Definition 2. The linear space $L_2^k(R^n)$ is defined as the class of those tempered distributions u on R^n, all of whose kth order derivatives are square integrable; in other words

$$L_2^k(R^n) = \{u \in S'(R^n) : D^\nu u \in L^2(R^n) \text{ for all } \nu \text{ with } |\nu| = k\}.$$

For this space, a semi-inner product is given by

$$\langle u, v\rangle_k = \sum_{|\nu|=k} c_\nu \int_{R^n} D^\nu u(x)\, \overline{D^\nu v(x)}\, dx,$$

where the positive constants c_ν are specified by

$$|\xi|^{2k} = \sum_{|\nu|=k} c_\nu \xi^{2\nu}. \tag{8}$$

Many properties about this space are discussed in [3]. We list only one that will be used later.

PROPOSITION 2. *The elements of $L_2^k(R^n)$ are continuous functions and there exists a linear map P on $L_2^k(R^n)$ with the following properties:*

(1) *Pu is in $\pi_{k-1}(R^n)$.*

(2) *If Ω is a unisolvent set for $\pi_{k-1}(R^n)$, then*

$$|Pu(x)| \leqslant C(1+|x|^{k-1})(\|u\|_{2,k}+\|u\|_{\Omega}),$$

where $\|u\|_{\Omega}$ denotes the maximum of u on Ω and C is a constant which depends on Ω but is independent of u.

(3) *If $Qu = u - Pu$, then Qu is continuous and satisfies*

$$|Qu(x)| \leqslant C(1+|x|^k)\,\|u\|_{2,k},$$

where C is independent of u.

DEFINITION 3. We say a function or a distribution is a polyharmonic cardinal spline if it is in one of the classes $SH_k(R^n)$ which is a subspace of $S'(R^n)$ whose elements f enjoy the following properties:

(i) f is in $C^{2k-n-1}(R^n)$ and

(ii) $\Delta^k f = 0$ on $R^n \backslash Z^n$.

THEOREM 1. *Given a sequence $v = \{v_{\mathbf{j}}\} \in l_2^k(Z^n)$, there is an element $f_{k,v}(x) = \sum_{\mathbf{j} \in Z^n} v_{\mathbf{j}} L_k(x-\mathbf{j})$ in $L_2^k(R^n) \cap SH_k(R^n)$ which interpolates v if and only if $v \in l_2^k(Z^n)$; and $\|f_{v,k}\|_{L_2^k} \leqslant C\,\|v\|_{l_2^k}$ where C is independent of v and $L_k(x)$ is defined by the Fourier transform*

$$\widehat{L_k}(\xi) = (2\pi)^{-n/2} \frac{|\xi|^{-2k}}{\sum_{\mathbf{j} \in Z^n} |\xi - 2\pi\mathbf{j}|^{-2k}}.$$

For more details about Proposition 2, Definition 3, and Theorem 1, see [2–4].

Applying T^{ν} to $f_{k,v}(x)$ implies

$$T^{\nu} f_{k,v}(x) = \sum_{\mathbf{j}} T^{\nu} v_{\mathbf{j}} L_k(x-\mathbf{j}). \tag{9}$$

Taking the Fourier transform on both sides of (9) yields

$$\prod_{j=1}^{n} (e^{i\xi_j}-1)^{\nu_j} \hat{f}_{k,v}(\xi) = \left(\sum_{\mathbf{j}} T^{\nu} v_{\mathbf{j}} e^{-i\langle \mathbf{j}, \xi \rangle}\right) \hat{L}_k(\xi). \tag{10}$$

We use this equation later.

The monotone property of $l_2^m(Z^n)$ with respect to m, that is, $l_2^m \subset l_2^{m+1}$, and Theorem 1 show that the sequence $v=(v_{\mathbf{j}}) \in l_2^k(Z^n)$ can be interpolated by a polyharmonic cardinal spline $f_{m,v} \in L_2^m(R^n) \cap SH_m(R^n)$ for $m>k$. It is natural to ask how $f_{m,v}$ behaves as m tends to infinity. In the cardinal spline case, this question was determined by I. J. Schoenberg (see [6]). In our case the limit still exists. We establish some theorems in the next section to determine the answer to our question. We conclude this section by defining a space which is an analogue to the analytic space defined in the cardinal spline case.

DEFINITION 4. $PW_\pi^k(R^n)=\{u \in S'(R^n) : \text{supp}(\hat{u}) \subset Q^n \text{ and } D^\nu u \in L^2(R^n)\}$ for all ν such that $|\nu|=k$.

If u is in this space, then $D^\nu u(x)=(2\pi)^{-n/2} \int_{Q^n} g_\nu(\xi)\, e^{-i\langle x,\xi\rangle}\, d\xi$ for some function $g_\nu \in L^2(Q^n)$ and $|\nu|=k$. Taking derivatives D^μ on both sides of this equation shows

$$D^{\nu+\mu}u=(2\pi)^{-n/2} \int_{Q^n} (-1)^{|\mu|}\, \xi^\mu g(\xi)\, e^{-i\langle x,\xi\rangle}\, d\xi.$$

Since the integrand is in $L^2(Q^n)$, we have

PROPOSITION 3. *$u \in PW_\pi^k(R^n)$ implies that $u \in PW_\pi^m(R^n)$ if $m>k$.*

3. THE INTERPOLATORY PROPERTY BETWEEN $l_2^k(Z^n)$ AND $PW_\pi^k(R^n)$

A relation between l_2^k and PW_π^k is established in this section for all integers k. When $k=0$, there is a known correspondence between the space l_2^0 and PW_π^0 which may be stated as follows. If

$$(v_{\mathbf{j}}) \in l_2^0 \tag{11}$$

then there is a unique function

$$F(x) \in PW_\pi^0 \tag{12}$$

such that

$$F(\mathbf{j})=v_{\mathbf{j}} \qquad \text{for all } \mathbf{j}. \tag{13}$$

Conversely, if (12) holds and we define $(v_{\mathbf{j}})$ by (13), then (11) holds.

We believe that there is a similar correspondence between l_2^k and PW_2^k for all integers k. In order to show this, three propositions are given below.

Proposition 4. *Given* $v = \{v_{\mathbf{j}}\} \in l_2^k(Z^n)$, *then for each* $\nu \in Z^n$ *with* $|\nu| = k$ *there exists a unique function* $g_\nu \in L^2(R^n)$ *such that*

$$(T^\nu v)_{\mathbf{j}} = (2\pi)^{-n/2} \int_{Q^n} g_\nu(\xi)\, e^{i\langle \xi, \mathbf{j}\rangle}\, d\xi.$$

For different ν *and* μ *we have*

$$e(\xi)^\mu g_\nu(\xi) = e(\xi)^\nu g_\mu(\xi), \tag{14}$$

where $e(\xi) = (e^{i\xi_1} - 1, ..., e^{i\xi_n} - 1)$.

Proof. Since $v \in l_2^k(Z^n)$, the following holds

$$\sum_{\mathbf{j} \in Z^n} |T^\nu v_{\mathbf{j}}|^2 < \infty$$

for any fixed ν with $|\nu| = k$. Then, the Riesz–Fischer theorem implies

$$T^\nu v_{\mathbf{j}} = (2\pi)^{-n/2} \int_{Q^n} g_\nu(\xi)\, e^{i\langle \xi, \mathbf{j}\rangle}\, d\xi,$$

where $g_\nu(\xi) = (2\pi)^{-n/2}\{\sum_{\mathbf{j}} T^\nu v_{\mathbf{j}} e^{-i\langle \mathbf{j}, \xi\rangle}\}\, \chi_{Q^n}(\xi)$.

Let $G_\nu(\xi) = \widehat{g_\nu}(-\xi)$, $G_\mu(\xi) = \widehat{g_\mu}(-\xi)$, and

$$g_{\nu,\mu}(\xi) = e(\xi)^\mu g_\nu(\xi) - e(\xi)^\nu g_\mu(\xi).$$

We have

$$G_\nu(\mathbf{j}) = T^\nu v_{\mathbf{j}}. \tag{15}$$

Consider the Fourier transform of $g_{\nu,\mu}$:

$$\begin{aligned}\widehat{g_{\nu,\mu}}(\mathbf{l}) &= (2\pi)^{-n/2} \int_{Q^n} g_{\nu,\mu}(\xi)\, e^{i\langle \xi, \mathbf{l}\rangle}\, d\xi \\ &= T^\mu G_\nu(\mathbf{l}) - T^\nu G_\mu(\mathbf{l}) \\ &= T^\mu T^\nu v_{\mathbf{l}} - T^\nu T^\mu v_{\mathbf{l}} \\ &= 0.\end{aligned}$$

Since this is true for all $\mathbf{l} \in Z^n$ and $g_{\nu,\mu}$ in $L^2(Q^n)$, we have $g_{\nu,\mu} = 0$. ∎

Proposition 5. *If* $u(x) \in PW_\pi^k$ *satisfies* $u(\mathbf{j}) = 0$ *for all* $\mathbf{j} \in Z^n$, *then* $u \equiv 0$.

Proof. By definition if $u \in PW_\pi^k$ then

$$D^\nu u(x) = (2\pi)^{-n/2} \int_{Q^n} g_\nu(\xi)\, e^{i\langle x, \xi\rangle}\, d\xi$$

for some $g_\nu(\xi) \in L^2(R^n)$ and $|\nu| = k$. This implies

$$T^\nu u(\mathbf{j}) = (2\pi)^{-n/2} \int_{Q^n} \prod_{j=1}^{n} \left(\frac{e^{i\xi_j} - 1}{i\xi_j}\right)^{\nu_j} g_\nu(\xi)\, e^{i\langle \mathbf{j}, \xi\rangle}\, d\xi.$$

Since $u(\mathbf{j}) = 0$, $\mathbf{j} \in Z^n$, we get $g_\nu(\xi) = 0$ a.e., for all ν such that $|\nu| = k$. Therefore

$$D^\nu u \equiv 0$$

for all ν with $|\nu| = k$, i.e., $u \in \pi_{k-1}(R^n)$. ∎

PROPOSITION 6. *Given a sequence $v = \{v_\mathbf{j}\}$, if $T^\nu v_\mathbf{j} = 0$ for all $\mathbf{j} \in Z^n$ and all ν such that $|\nu| = k$, then there exists a function $f(x) \in \pi_{k-1}(R^n)$ such that $f(\mathbf{j}) = v_\mathbf{j}$ for all $\mathbf{j} \in Z^n$.*

Proof. The proof proceeds by induction on dimensions. First, we know that the conclusion follows in R^1. Suppose that it is correct in R^{n-1}. Let $\nu = (k, 0, ..., 0)$, then we have

$$T^\nu v_\mathbf{j} = T_1^k v_\mathbf{j} = 0. \tag{16}$$

Fix $\mathbf{j}_1 = (j_2, ..., j_n)$, then (16) implies, by the result in R^1, that there exists a function $f_{j_2 \cdots j_n}(x_1) = \sum_{i=0}^{k-1} C_i(j_2 \cdots j_n)\, x_1^i$ such that

$$f_{j_2 \cdots j_n}(j_1) = v_\mathbf{j}.$$

Claim. If $\nu = (i, \mu)$, $\mu = (\nu_2, ..., \nu_n)$, and $|\mu| = k - i$, $i = 0, ..., k-1$, then

$$T^\mu C_i(\mathbf{j}_1) = 0.$$

The proof of the claim is a simple application of induction on i starting with $i = k - 1$.

Since $\mathbf{j}_1 \in R^{n-1}$, by induction, we have a function $f_i(x_2, ..., x_n) \in \pi_{k-i}(R^{n-1})$ such that $f_i(j_2, ..., j_n) = C_i(\mathbf{j}_1)$. Let $f(x) = \sum_{i=0}^{k-1} f_i(x_2, ..., x_n)\, x_1^i$. The proposition follows. ∎

THEOREM 2. *Given $v = \{v_\mathbf{j}\} \in l_2^k(R^n)$, there exists a unique function $f \in PW_\pi^k(R^n)$ such that*

(1) $f(\mathbf{j}) = v_\mathbf{j}$, $\mathbf{j} \in Z^n(R^n)$,

(2) $\sum_{|\nu| = k} \int_{R^n} |D^\nu f(x)|^2\, dx \leqslant (\pi/2)^{2k}\, \|v\|_{2,k}^2$,

(3) $\sum_{|\nu|=k} |D^\nu f(x)|^2 \leqslant (2\pi)^n (\pi/2)^{2k} \|v\|_{2,k}^2$ *for all* x *in* R^n.

Proof. Given an index ν with $|\nu|=k$, by Proposition 4, there exists a function $g_\nu(\xi) \in L^2(Q^n)$ such that

$$(T^\nu v)_{\mathbf{j}} = (2\pi)^{-n/2} \int_{Q^n} g_\nu(\xi)\, e^{i\langle \mathbf{j}, \xi\rangle}\, d\xi. \tag{17}$$

For simplicity we assume that $g_\nu(\xi)=0$ on R^n/Q^n.

Define a new function $w(\xi)=0$ by

$$w(\xi) = \frac{(-1)^k\, |\xi|^{2k}}{\sum_{|\nu|=k} c_\nu e(\xi)^{2\nu}} \sum_{|\nu|=k} c_\nu e(\xi)^\nu g_\nu(\xi),$$

where c_ν and $e(\xi)^\nu$ are defined as above. Let $W(x)=\hat{w}(-x)$, then $W(x)$ is a tempered distribution. Therefore, there exists a solution which is also a tempered distribution to the equation

$$\Delta^k F(x) = W(x).$$

Choose any solution $F(x)$, then we have

$$(-1)^k\, |\xi|^{2k}\, \hat{F}(\xi) = w(\xi),$$

that is,

$$|\xi|^{2k}\, \hat{F}(\xi) = \frac{|\xi|^{2k}}{\sum_{|\nu|=k} c_\nu e(\xi)^{2\nu}} \sum_{|\nu|=k} c_\nu e(\xi)^\nu g_\nu(\xi), \qquad \xi \in R^n.$$

Choose μ such that $|\mu|=k$, multiply $e(\xi)^\mu$ on both sides of this equation, and use Proposition 4, then we get

$$e(\xi)^\mu\, \hat{F}(\xi) = g_\mu(\xi), \qquad \xi \in R^n. \tag{18}$$

Observe

$$\begin{aligned} T^\mu F(\mathbf{j}) &= (2\pi)^{-n/2} \int_{Q^n} \widehat{T^\mu F}(\xi)\, e^{i\langle \mathbf{j}, \xi\rangle}\, d\xi \\ &= (2\pi)^{-n/2} \int_{Q^n} e(\xi)^\mu\, \hat{F}(\xi)\, e^{i\langle \mathbf{j}, \xi\rangle}\, d\xi \\ &= (2\pi)^{-n/2} \int_{Q^n} g_\mu(\xi)\, e^{i\langle \mathbf{j}, \xi\rangle}\, d\xi \\ &= T^\mu v_{\mathbf{j}}. \end{aligned} \tag{19}$$

Multipling $(i\xi)^{\mu}$ on both sides of (18) implies

$$(i\xi)^{\mu}\,\hat{F}(\xi)=\frac{(i\xi)^{\mu}}{e(\xi)^{\mu}}\,g_{\mu}(\xi), \qquad \text{if} \quad \xi\in R^{n}. \tag{20}$$

Since $|t/\sin t|\leqslant \pi/2$ for $t\in(-\pi/2,\pi/2)$,

$$\left|\frac{(i\xi)^{\mu}}{e(\xi)^{\mu}}\right|\,|g_{\mu}(\xi)|=\prod_{j=1}^{n}\left|\frac{\xi_j}{2\sin(\xi_j/2)}\right|^{\mu_j}|g_{\mu}(\xi)|\leqslant\left(\frac{\pi}{2}\right)^{k}|g_{\mu}(\xi)|. \tag{21}$$

Therefore, the following holds by taking the inverse Fourier transform on both sides of (20)

$$D^{\mu}F(x)=(2\pi)^{-n/2}\int_{Q^n}\frac{(i\xi)^{\mu}}{e(\xi)^{\mu}}\,g_{\mu}(\xi)\,e^{i\langle x,\xi\rangle}\,d\xi. \tag{22}$$

This shows that $F\in PW^{k}_{\pi}$.

Equation (19) together with Proposition 6 implies the existence of a unique function $P(x)\in\pi_{k-1}$ such that

$$F(\mathbf{j})+P(\mathbf{j})=v_{\mathbf{j}}.$$

Let $f=F+P$, then f is the function for which we are looking. The uniqueness follows from Proposition 5. Recall

$$D^{\nu}f(x)=\frac{1}{(2\pi)^{n/2}}\int_{Q^n}\frac{(i\xi)^{\nu}}{e(\xi)^{\nu}}\,g_{\nu}(\xi)\,e^{i\langle x,\xi\rangle}\,d\xi.$$

Plancherel's theorem and (20) imply

$$\begin{aligned}\sum_{|\nu|=k}\int_{R^n}|D^{\nu}f(\xi)|^2\,d\xi&=\sum_{|\nu|=k}\int_{Q^n}\left|\prod_{j=1}^{n}\left(\frac{i\xi_j}{e^{i\xi_j}-1}\right)^{\nu_j}\right|^2|g_{\nu}(\xi)|^2\,d\xi\\&\leqslant\left(\frac{\pi}{2}\right)^{2k}\sum_{|\nu|=k}\int_{Q^n}|g_{\nu}(\xi)|^2\,d\xi\\&=\left(\frac{\pi}{2}\right)^{2k}\sum_{|\nu|=k}\sum_{\mathbf{j}\in Z^n}|\widehat{g_{\nu}}(\mathbf{j})|^2\\&=\left(\frac{\pi}{2}\right)^{2k}\|\mathbf{v}\|^2_{2,k}.\end{aligned}$$

Again using Plancherel's theorem gives us

$$\begin{aligned}
\sum_{|\nu|=k} |D^\nu f(x)|^2 &= \sum_{|\nu|=k} \left| \int_{Q^n} g_\nu(\xi) \prod_{j=1}^n \left(\frac{i\xi_j}{e^{i\xi_j}-1} \right)^{\nu_j} e^{i\langle x,\xi\rangle}\, d\xi \right|^2 \\
&\leqslant \left(\frac{\pi}{2}\right)^{2k} \sum_{|\nu|=k} \left\{ \int_{Q^n} |g_\nu(\xi)|\, d\xi \right\}^2 \\
&\leqslant (2\pi)^n \left(\frac{\pi}{2}\right)^{2k} \sum_{|\nu|=k} \int_{Q^n} |g_\nu(\xi)|^2\, d\xi \\
&= (2\pi)^n \left(\frac{\pi}{2}\right)^{2k} \sum_{|\nu|=k} \sum_{\mathbf{j}\in Z^n} |\widehat{g_\nu}(\mathbf{j})|^2 \\
&= (2\pi)^n \left(\frac{\pi}{2}\right)^{k} \sum_{|\nu|=k} \sum_{\mathbf{j}} |T^\nu v_{\mathbf{j}}|^2 \\
&= (2\pi)^n \left(\frac{\pi}{2}\right)^{2k} \|v\|_{2,k}^2. \quad \blacksquare
\end{aligned}$$

4. The Convergence of Polyharmonic Cardinal Splines

Many properties of the fundamental spline $L_m(x)$ will be used throughout the proofs of the theorems in this section. Some of these properties are listed below.

(1) $\sum_{\mathbf{j}} \hat{L}_m(\xi + 2\pi\mathbf{j}) = (2\pi)^{-n/2}$, $\xi \in R^n$.

(2) $\lim_{m\to\infty} \hat{L}_m(\xi) = (2\pi)^{-n/2} \chi_{Q^n}(\xi)$, a.e. for $\xi \in R^n$.

(3) $|\hat{L}_m(\xi)| \geqslant C > 0$ if $m > k$ and $\xi \in Q^n$, where C is a constant independent of m, ξ and k.

(4) $\sum_{\mathbf{j}} |\xi + 2\pi\mathbf{j}|^{2k} \hat{L}_m^2(\xi + 2\pi\mathbf{j}) = (2\pi)^{-n/2} |\xi|^{2k} (\hat{L}_m^2(\xi)/\hat{L}_{2m-k}(\xi))$.

(5) If $\tilde{g}_\nu(\xi)$ is the periodic extension of $g_\nu(\xi)$ in Q^n, then

$$\int_{R^n} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} |\tilde{g}_\nu(\xi)|^2 \hat{L}_m^2(\xi)\, d\xi = (2\pi)^{-n/2} \int_{Q^n} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} \frac{\hat{L}_m^2(\xi)}{\hat{L}_{2m-k}(\xi)} |g_\nu(\xi)|^2\, d\xi.$$

The first two properties are easily verified. We start with the proof of property (3). Since $\hat{L}_m(\xi)$ is symmetric, we may assume $\xi \in (0, \pi)^n$. Observe that $|t/(t+2\pi_j)|$ is an increasing function in $(0, \pi)$ for all $j \in Z$. Then

$$\sup_{t\in(0,\pi)} \left| \frac{t}{t+2\pi j} \right| \leqslant \frac{1}{|1+2j|} \leqslant 1, \tag{23}$$

and

$$\sup_{\xi \in Q^n} \frac{|\xi|}{|\xi + 2\pi \mathbf{j}|} \leqslant 1 \qquad \text{for all} \quad \mathbf{j} \in Z. \tag{24}$$

Consider

$$\begin{aligned}
\frac{(2\pi)^{-n/2}}{\hat{L}_m(\xi)} &= 1 + \sum_{\mathbf{j} \neq 0} \frac{|\xi|^{2m}}{|\xi + 2\pi \mathbf{j}|^{2m}} \leqslant 1 + \sum_{\mathbf{j} \neq 0} \frac{|\xi|^{2k}}{|\xi + 2\pi \mathbf{j}|^{2k}} \\
&= 1 + \sum_{\mathbf{j} \neq 0} \left(\frac{\sum_{i=1}^n (\xi_i + 2\pi j_i)^2 (\xi_i / \xi_i + 2\pi j_i))^2}{|\xi + 2\pi \mathbf{j}|^2} \right)^k \\
&\leqslant 1 + \sum_{\mathbf{j} \neq 0} \left(\sum_{i=1}^n \frac{1}{(1 + 2j_i)^2} \frac{(\xi_i + 2\pi j_i)^2}{|\xi + 2\pi \mathbf{j}|^2} \right)^k \\
&\leqslant 1 + \sum_{\mathbf{j} \neq 0} \left(\sum_{i=1}^n \frac{1}{(1 + 2\pi j_i)^2} \right)^k < \infty.
\end{aligned}$$

We get the first inequality by using (23). This completes the proof of property (3).

To prove property (4), observe

$$\begin{aligned}
\sum_{\mathbf{j}} |\xi + 2\pi \mathbf{j}|^{2k} \hat{L}_m^2(\xi + 2\pi \mathbf{j}) &= (2\pi)^{-n} \frac{\sum_{\mathbf{j}} (1/|\xi + 2\pi \mathbf{j}|^{4m-2k})}{(\sum_{\mathbf{j}} (1/|\xi + 2\pi \mathbf{j}|^{2m}))^2} \\
&= (2\pi)^{-n/2} |\xi|^{2k} \frac{\hat{L}_m^2(\xi)}{\hat{L}_{2m-k}(\xi)}.
\end{aligned}$$

Property (5) is a simple application of property (4) because

$$\begin{aligned}
&\int_{R^n} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} |\tilde{g}_v(\xi)|^2 \hat{L}_m^2(\xi)\, d\xi \\
&\quad = \sum_{\mathbf{j} \in Z^n} \int_{Q^n + 2\pi \mathbf{j}} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} |\tilde{g}_v(\xi)|^2 \hat{L}_m^2(\xi)\, d\xi \\
&\quad = \int_{Q^n} \frac{1}{|e(\xi)|^{2k}} |g_v(\xi)|^2 \sum_{\mathbf{j} \in Z^n} |\xi + 2\pi \mathbf{j}|^{2k} \hat{L}_m^2(\xi + 2\pi \mathbf{j})\, d\xi \\
&\quad = (2\pi)^{-n/2} \int_{Q^n} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} |g_v(\xi)|^2 \frac{\hat{L}_m^2(\xi)}{\hat{L}_{2m-k}(\xi)}\, d\xi.
\end{aligned}$$

Now we are ready to show the main theorem.

THEOREM 3. *Given* $v = \{v_j\} \in l_2^k$, *then the unique function* $f_{m,v}$ *in* $L_2^m(R^n) \cap SH_m(R^n)$ *given by Theorem* 1 *has the property*

$$\lim_{m \to \infty} \|f_{m,v} - f_v\|_{2,k} = 0,$$

where $f_v \in PW_\pi^k(R^n)$ *is given by Theorem* 2.

Proof. Recall from (10), (17), and (18)

$$e(\xi)^\nu \hat{f}_{m,v}(\xi) = (2\pi)^{n/2} \hat{g}_\nu(\xi) \hat{L}_m(\xi), \tag{25}$$

$$e(\xi)^\nu \hat{f}_v(\xi) = g_\nu(\xi), \tag{26}$$

$$g_\nu(\xi) = \tilde{g}_\nu(\xi)\, \chi(\xi).$$

Computing $\sum_{|\nu|=k} c_\nu [(25)-(26)]^2$ implies

$$|e(\xi)|^{2k} \, |\hat{f}_{m,v}(\xi) - \hat{f}_v(\xi)|^2 \leqslant \sum_{|\nu|=k} c_\nu |\tilde{g}_\nu(\xi)|^2 \, |(2\pi)^{n/2} \hat{L}_m(\xi) - \chi_{Q^n}(\xi)|^2, \tag{27}$$

where c_ν is defined in (8).

Consider

$$\begin{aligned}
&\int_{R^n} |D^\nu f_{m,v}(\xi) - D^\nu f_v(\xi)|^2 \, d\xi \\
&\quad = \int_{R^n} |\xi^\nu|^2 \, |\hat{f}_{m,v}(\xi) - \hat{f}_v(\xi)|^2 \, d\xi \\
&\quad \leqslant \int_{R^n} |\xi|^{2k} \, |\hat{f}_{m,v}(\xi) - \hat{f}_v(\xi)|^2 \, d\xi \\
&\quad = \int_{R^n} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} \, |e(\xi)|^{2k} \, |\hat{f}_{m,v}(\xi) - \widehat{f_v}(\xi)|^2 \, d\xi \\
&\quad \leqslant \sum_{|\nu|=k} c_\nu \int_{R^n} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} \, |\tilde{g}_\nu(\xi)|^2 \, |(2\pi)^{n/2} \hat{L}_m(\xi) - \chi_{Q^n}(\xi)|^2 \, d\xi.
\end{aligned}$$

Because $\lim_{m \to \infty} \hat{L}_m(\xi) = (2\pi)^{-n/2} \chi_{Q^n}(\xi)$ holds almost everywhere in R^n, Lebesgue's dominated theorem implies

$$\lim_{m \to \infty} \int_{Q^n} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} \, |g_\nu(\xi)|^2 \, |(2\pi)^{n/2} \hat{L}_m(\xi) - 1|^2 \, d\xi = 0.$$

On the other hand

$$\int_{R^n - Q^n} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} |\tilde{g}_v(\xi)|^2 \hat{L}_m^2(\xi)\, d\xi$$

$$= \left(\int_{R^n} - \int_{Q^n}\right) \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} |\tilde{g}_v(\xi)|^2 \hat{L}_m^2(\xi)\, d\xi$$

$$= \int_{Q^n} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} \frac{\hat{L}_m^2(\xi)}{\hat{L}_{2m-k}(\xi)} [(2\pi)^{-n/2} - \hat{L}_{2m-k}(\xi)]\, |g_v(\xi)|^2\, d\xi.$$

By Lebesgue's dominated theorem again, we have

$$\lim_{m \to \infty} \int_{R^n - Q^n} \frac{|\xi|^{2k}}{|e(\xi)|^{2k}} |\tilde{g}_v(\xi)|^2 \hat{L}_m^2(\xi)\, d\xi = 0.$$

We have just shown that for any v such that $|v| = k$

$$\lim_{m \to \infty} \|D^v f_{m,v} - D^v f_v\|_2 = 0. \tag{28}$$

Finally, we get

$$\lim_{m \to \infty} \|f_{m,v} - f_v\|_{2,k} = 0$$

for every element of $l_2^k(Z^n)$. ∎

We derive the following directly from Theorem 3 together with the monotone property of $l_2^m(Z^n)$.

COROLLARY 1. *Under the same conditions as above, we have*

$$\lim_{m \to \infty} \|f_{m,v} - f_v\|_{2,k'} = 0,$$

where $k' > k$.

THEOREM 4. *Under the same condition as the theorem above, the relation*

$$\lim_{m \to \infty} f_{m,v}(x) = f_v(x)$$

holds uniformly in x *on any compact subset of* R^n.

Proof. Choose any fixed pair of complementary projections P and Q whose existence are guaranteed by Proposition 2. Then for any element u

of $L_2^k(R^n)$, we have $u = Pu + Qu$. Let $u = f_{m,v} - f_v$, and consider Pu and Qu separately. By Proposition 2 we have

$$|Qu(x)| \leqslant C(1 + |x|^k)\, \|u\|_{2,k},$$

i.e.,

$$|Q(f_{m,v} - f_v)(x)| \leqslant C(1 + |x|^k)\, \|f_{m,v} - f_v\|_{2,k}.$$

Then the relation

$$\lim_{m \to \infty} Q(f_{m,v} - f_v)(x) = 0$$

holds uniformly on any compact subset of R^n.

On the other hand, let Ω be a unisolvent set of Z^n, then $\|f_{m,v} - f_v\|_\Omega = 0$. By Proposition 2 we have

$$|P(f_{m,v} - f_v)(x)| \leqslant C(1 + |x|^{k-1})(\|f_{m,v} - f_v\|_{2,k} + \|f_{m,v} - f_v\|_\Omega).$$

This implies

$$|P(f_{m,v} - f_v)(x)| \leqslant C(1 + |x|^{k-1})\, \|f_{m,v} - f_v\|_{2,k}.$$

Therefore, the following holds

$$\lim_{m \to \infty} P(f_{m,v} - f_v)(x) = 0$$

uniformly on any compact subset of R^n. This completes the proof. ∎

COROLLARY 2. *Under the same conditions as above we have*

$$\lim_{m \to \infty} D^\nu f_{m,v}(x) = D^\nu f_v(x)$$

uniformly for all v *on any compact subset of* R^n.

Proof. Let $g_{m,v} = D^\nu f_{m,v}$ and $F_v = D^\nu f_v$, apply Corollary 1 and Theorem 4, and the result follows. ∎

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